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THE PROBLEM OF MEASUREMENT FEEDBACK CONTROL†

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The deterministic problem of constructing synthesized control strategies from the results of available observations under unknown but bounded disturbances subjected in advance to hard bounds on the uncertain parameters is considered. The suggested solutions rely on methods of dynamic programming and set-valued analysis and are formulated in terms of the Hamiltonian formalism. It is shown that the problem can be separated into two – a finite-dimensional problem of estimation and an infinite-dimensional control problem.

The central topic among the problems of control synthesis is the investigation of systems with incomplete information. It is directed towards constructing the best or better control laws in some appropriate sense and also to indicating how the level of uncertainty and the "amount of information" given in advanced or arriving on line would affect the values of the objective functions or some other estimates of its performance levels.

The leading role in initiating these topics belongs to N. N. Krasovskii, who suggested both game-theoretic and stochastic approaches [1–5]. A vast literature devoted to stochastic formulations of such problems, where, in particular, the principle of separating the overall solution into independent solutions of problems of observation and control has been proposed [6–9]. New interpretations of the solutions of problems of control with incomplete information have been proposed within the framework of the so-called H_∞ theory [10–12].

In this paper we consider the deterministic problem of constructing synthesized control strategies from the results of available observations under conditions of unknown but bounded disturbances subjected in advance to hard bounds on the uncertain parameters. This paper continues previous investigations discussed in [13–15].

1. THE BASIC PROBLEM. PRELIMINARY FORMULATIONS

We will first indicate the formulation of the general approach to the problem of synthesizing controls from the results of measurements (observations).

Consider the n-dimensional system

$$dx/dt = f_1(t, x, u) + f_2(t, x, \upsilon) \tag{1.1}$$

where the functions $f_1(t, x, u)$ and $f_2(t, x, \upsilon)$ are continuous in all the variables and such that their sum satisfies standard conditions of uniqueness and extendability of the solutions of Eq. (1.1) throughout over the finite interval $[t_0, t_1]$ for any initial condition $x^0 \in \mathbf{R}^n$, and also for any admissible control $u(t)$ and disturbances $\upsilon(t)$, restricted by geometrical (Chebyshev) constraints

$$u(t) \in \mathcal{P}(t), \quad \upsilon(t) \in \mathcal{Q}(t), \quad x(t_0) \in \mathcal{X}_0 \tag{1.2}$$

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for all $t \in [t_0, t_1]$ [5, 16]. Here $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ are multivalued functions with values in the set of compacta of spaces $\mathbf{R}^p$ and $\mathbf{R}^q$ respectively, continuous in the Hausdorff metric, and the set $\mathcal{X}_0$ is a compactum. The pair $\{t_0, \mathcal{X}_0\}$ will be called the "initial position" of the system.

We shall also assume that the sets $\mathcal{F}_1(t, x) = f_1(t, x, \mathcal{P}(t))$ and $\mathcal{F}_2(t, x) = f_2(t, x, \mathcal{Q}(t))$ are convex and compact. Then, in view of the above properties, the set-valued functions $\mathcal{F}_1$ and $\mathcal{F}_2$ will be Hausdorff-continuous in all the variables.

Current information on the vector x is given by means of observations, which are obtained by virtue of the measurement equation

$$y(t) = g(t, x) + w \tag{1.3}$$

where $y(t) \in \mathbf{R}^r$ is the available measurement and $w(t)$ is the unknown disturbance, information on which is restricted by the specification of the limitation

$$w(t) \in \mathcal{R}(t), \quad t \in [t_0, t_1] \tag{1.4}$$

The function $g(t, x)$ is assumed to be continuous in all the variables, while the properties of the function $\mathcal{R}(t)$ are similar to those of $\mathcal{P}(t)$ and $\mathcal{Q}(t)$.

Knowing the initial position $\{t_0, \mathcal{X}^0\}$, the functions $f_1(t, x, u, \upsilon)$, $f_2(t, x, u, \upsilon)$ and $g(t, x)$, the form of the control $u[t]$ $(t \in [t_0, \tau])$, the set-valued functions $\mathcal{Q}(t)$ and $\mathcal{R}(t)$ and available measurements $y_\tau(\sigma) = y(\tau + \sigma)$ $(\sigma \in [t_0 - \tau, 0])$, one may construct the *information set* $\mathcal{X}(\tau, y_\tau(\cdot)) = \mathcal{X}(\tau, \cdot)$ of system (1.1)–(1.4), consistent with its parameters and with the measurements obtained. Thus, the actual *position* of the system may be taken as the pair $\{\tau, \mathcal{X}(\tau, \cdot)\}$. The use of the information sets and the description of their properties are the subject of the *theory of guaranteed estimation* [1, 13, 14, 17, 18].

The problem of control synthesis to be considered will consist of specifying the control strategy $u(\tau, \mathcal{X}(\tau, \cdot))$ $(\tau \in [t_0, t_1])$, constructed from the results of observations, which, for any initial position $\{t_0, \mathcal{X}^0\}$, would bring the vector $x(t_1)$ into a preassigned neighbourhood of a given target set – a compactum $\mathcal{M}$ – *whatever* the unknown disturbances.

At the same time the class of admissible strategies $\mathcal{U} = \{u(\tau, \mathcal{X}(\tau, \cdot))\}$ will have to ensure the existence and prolongability of the solutions of equation (the differential inclusion) (1.1) with $u = u(\tau, \mathcal{X}(\tau, \cdot))$ $(\tau \in [t_0, t_1])$.

Since the position $\{t_0, \mathcal{X}^0\}$ is arbitrary, the solution obtained must also hold for any actual position $\{\tau, \mathcal{X}(\tau, \cdot)\}$ $(\tau \in [t_0, t_1])$, if the latter is taken as the initial one.

Thus the solution of the overall problem will consist of combining the processes of observation and control. Suppose that over the interval $[t_0, \tau)$ the control $u^*(t)$ and the observation $y^*(t)$ have been realized. As a preliminary move the problem under consideration may be explained with the aid of the functional

$$\mathcal{V}(t_0, \mathcal{X}^0 | u^*(t), y^*(t), t_0 \le t \le \tau) = \min_u \max_{\zeta(\cdot)} \Big\{ -d^2(x(t_0), \mathcal{X}^0) -$$

$$- \int_{t_0}^{\tau} d^2(y^*(t) - g(t, x(t)), \mathcal{R}(t))dt - \int_{\tau}^{t_1} d^2(\xi(t), \mathcal{R}(t))dt + d^2(x(t_1), \mathcal{M}) \Big\} \tag{1.5}$$

Here $d^2(x, \mathcal{X}) = \min\{(x - z, x - z) | z \in \mathcal{X}\}$ is the square of the Euclidian distance between x and set $\mathcal{X}$, while $x(t_0)$, $x(t_1)$ are the ends of the trajectory $x(t)$ of system (1.1), $\zeta(\cdot) = \{x(t_0), \upsilon(\cdot), \xi(\cdot)\}$. Note that $\upsilon = \upsilon(t)$ is considered within the interval $[t_0, t_1]$, while $u = u(t, \mathcal{X}(t, \cdot))$ and $\xi = \xi(t)$ are considered within the interval $(\tau, t_1]$. Also note that over the interval $[t_0, \tau]$, where the observation $y^*(t)$ is known, the realization of the observation disturbances is $\xi^*(t) = y^*(t) - g(t, x(t))$.

The functional (1.5) has to be maximized over $\zeta(\cdot)$ and minimized over u in the class of the strategies described further.

Below, in Sections 3–5, a more rigorous formulation and interpretation of the problem will be given. We wish to emphasize, however, that from the preliminary description we can already conclude that the overall problem $E - C$ may be separated into two, namely, into problem E on guaranteed estimation (calculation of the actual position of the system) and problem C on constructing precisely the synthesized controls which realize the preassigned goals of the controlled process (as functionals of the actual position of the system).

2. GUARANTEED ESTIMATION OF THE STATE OF THE SYSTEM

We shall consider problem E in two versions – Problem E and Problem E_0.

Problem E. Suppose we are given system (1.1), (1.2) and measurement equation (1.3), (1.4). Suppose we also know the starting position $\{t_0, \mathcal{X}^0\}$, the measurements $y^*_\tau(\cdot)$, and the realization of the control $u = u^*(t)$ $(t \in [t_0, \tau))$. It is required to specify *the information set* $\mathcal{X}(\tau, \cdot)$ of such states $x(\tau)$ generated by system (1.1), which are consistent with $y^*(t)$, as well as with constraints (1.1)–(1.4) and realization $u^*(t)$.

"The information set" $\mathcal{X}[\tau] = \mathcal{X}(\tau, \cdot)$ is a *guaranteed estimate* of the vector $x(\tau)$. It includes the unknown actual vector $x(\tau)$ for system (1.1). The last fact indicates that the *actual state* of the overall system with incomplete measurements and $t \geq t_0$, may be taken as the pair $\{t, \mathcal{X}[t]\}$. (An equivalent definition of the actual state may be served also by the pair $\{t, y_t(\cdot)\}$ $(y_t(\cdot) = y(t + \sigma), \sigma \in [t_0 - t, 0])$.)

To solve the basic Problem E and C one should first describe the evolution of sets $\mathcal{X}[\tau] = \mathcal{X}(\tau, \cdot)$ in time. This may be done by solving the following alternative problem.

Problem E_0. Suppose we are given the starting position $\{t_0, \mathcal{X}^0\}$, the measurements $y^*_\tau(\cdot)$, and the control realization $u^*(t)$ with $t \in [t_0, \tau)$. It is required to obtain

$$-V(\tau, x) = \max_v \Bigg\{ -d^2(x[t_0], \mathcal{X}^0) - \int_{t_0}^{\tau} d^2(y^*(t) - g(t, x), \mathcal{R}(t))dt \,|\, x[t_0] \in \mathbf{R}^n, \quad v(t) \in \mathcal{Q}(t), \quad t \in [t_0, \tau] \Bigg\} \tag{2.1}$$

under condition $x[\tau] = x$, by virtue of system (1.1).

The initial condition for calculating $V(t, x)$ will be given by the equality $V(t_0, x) = d^2(x, \mathcal{M})$.

Here $x[t] = x(t, \tau, x)$ $(t \leq \tau)$ denotes the backward trajectory, emanating from the point $\{\tau, x\}$ due to system (1.1).

Remark 2.1. If we represent functional (1.5) as the sum of two parts: the first, defined for $[t_0, \tau]$, and the second, defined for $(\tau, t_1]$, then functional (2.1) will be precisely the first part, reflecting the problem of state estimation (the calculation of the actual state of the system).

Function $V(\tau, x)$ will be further referred to as the "information state" of system (1.1)–(1.4).

Lemma 2.1. The information set $\mathcal{X}[\tau]$ is the following level set of the information state – the function $V(\tau, x)$

$$\mathcal{X}[\tau] = \{x : V(\tau, x) \leq 0\}) \tag{2.2}$$

This implies that the actual state, namely *the position* of the system may be defined not only as one of the pairs $\{\tau, \mathcal{X}[\tau]\}$, $\{\tau, y_\tau(\cdot)\}$, as indicated above, but also as the pair $\{\tau, V(\tau, \cdot)\}$. In what follows we will mostly use the last definition, without rejecting the other two when we need to explain the essential steps of the solution procedure. Using the last definition is convenient since the position $\{\tau, V(\tau, \cdot)\}$ may be represented in more conventional terms of partial differential equations.

For the function $V(\tau, x)$ we further introduce the more detailed notation $V(\tau, x) = V(\tau, x \,|\, V(t_0, \cdot))$, emphasizing the dependence of this function on the initial value $V(t_0, \cdot)$.

Lemma 2.2. The following property holds

$$V(\tau, x \,|\, V(t_0, \cdot)) = V(\tau, x \,|\, V(t, \cdot \,|\, V(t_0, \cdot))), \quad t_0 \leq t \leq \tau \tag{2.3}$$

Remark 2.2. Formula (2.3) expresses "the principle of optimality" for the problem of guaranteed state estimation as taken in version E_0. It reflects the *semigroup property* of the mapping $V(\tau, x \,|\, V(t_0, \cdot))$.

Formula 2.3 yields the "forward" Hamilton–Jacobi–Bellman (HJB) partial differential equation for the function $V(t, x)$. This equation is formally derived following the standard schemes of the theory of dynamic programming [8, 21].

The equation under discussion is as follows:

$$\frac{\partial V}{\partial t} = -\max_{v}\left\{\left(\frac{\partial V}{\partial x}, f_1(t, x, u^*(t)) + f_2(t, x, v)\right) - \right.$$

$$\left. - d^2(y^*(t) - g(t, x), \mathcal{R}(t)) \mid v(t) \in \mathcal{Q}(t)\right\}, \quad t \in [t_0, \tau] \tag{2.4}$$

with the boundary condition

$$V(t_0, x) = d^2(x, \mathcal{X}^0) \tag{2.5}$$

Function $V(t, x)$ may be non-smooth, hence Eq. (2.4) under condition (2.5) may not have a classical solution. The solution of this equation should then be understood in a generalized sense, for example, as a "viscosity" solution [19–22] or a "minmax" solution [23]. In the general case it may be defined in terms of generalized derivatives, Dini subdifferentials or equivalent notions [24].

Theorem 2.1. If, with $u = u^*(t), y = y^*(t)$ $(t \in [t_0, \tau])$, Eq. (2.4) under condition (2.5) has a generalized (viscosity) solution $V(t, x)$, then the following relation holds

$$\mathcal{X}[\tau] = \{x : V(\tau, x) \leq 0\} \tag{2.6}$$

If Eq. (2.4) under condition (2.5) has a classical solution, then, of course, it is identical with the generalized solution.

The existence of a generalized solution of Eq. (2.4) under condition (2.5) is necessary and sufficient for a solution of Problem E_0 to exist. In this case the mapping $V[t] = V(t, \cdot) = V(t, \cdot \mid V(t_0, \cdot))$ satisfies an evolution equation of the type

$$\frac{\partial V(t, \cdot)}{\partial t} = \Phi(t, V_x(t, \cdot), \cdot \mid u^*(t), y^*(t)) \tag{2.7}$$

which is nothing else than Eq. (2.4), namely, its right-hand side is identical with the right-hand side of Eq. (2.4).

Instead of solving the HJB equation (2.4) under condition (2.5), defining the information set $\mathcal{X}[\tau]$ due to Lemma 2.1, another description of this set is possible.

We will derive an alternative HJB equation with additional assumptions on the smoothness and convexity of the constraints. Suppose we are given proper continuously differentiable functions $\varphi_0(x)$ and $\varphi(t, x)$, convex in x. Then the initial set and the constraint on the disturbance ξ in measurement equation (1.3), may be represented in the form

$$\mathcal{X}^0 = \{x : \varphi_0(x) \leq 1\}, \quad \mathcal{R}(t) = \{x : \varphi(t, y(t) - g(t, x(t))) \leq 1\}$$

The second inequality defines restrictions on the phase coordinates of system (1.1).

In particular, the functions φ_0 and φ may be specified by the equalities

$$\varphi_0(x^0) = d^2(x^0, \mathcal{X}^0) + 1, \quad \varphi(t, x(t)) = d^2(g(t, x(t)), Y(t)) + 1$$

where the set-valued mapping $Y(t) = y(t) - \mathcal{R}(t)$, where the mapping $\mathcal{R}(t)$ is Hausdorff-continuous and the function $y(t)$ is right-continuous. Such types of constraints are quite common for both linear and non-linear systems. The use of these constraints in the present text will enable us to explain the solutions in a more transparent form.

Suppose that over the interval $[t_0, t]$ the realizations $u^*(\cdot)$ and $y^*(\cdot)$ of the control u and observation y are known.

Consider the new value function

$$V^{(1)}[\tau, x] = V^{(1)}(\tau, x; u^*(\cdot), y^*(\cdot)) =$$

$$= \min_{u}\{\varphi_0(x[t_0]) \mid x[\tau] = x, \ \varphi(t, y(t) - g(t, x[t])) \leq 1, \ t \in [t_0, \tau]\} \tag{2.8}$$

Putting

$$\frac{dV(t,x)}{dt} = \frac{\partial V(t,x)}{\partial t} + \left(\frac{\partial V(t,x)}{\partial x}, f(t,x,u^*,v)\right) = V_t + \mathcal{H}(t,x,V_x,u^*,v)$$

consider the equation

$$V_t^{(1)} + \max_v \{\mathcal{H}(t,x,V_x^{(1)},u^*,v) \,|\, v \in \mathcal{Q}_\varphi(t)\} = 0, \quad V^{(1)}(t_0,x) = \varphi_0(x)$$

$$\mathcal{Q}_\varphi(t) = \begin{cases} \mathcal{Q}(t), & \theta(t) < 1 \\ \mathcal{Q}(t) \cap \{v : d\theta(t)/dt \le 0 \,|\, u = u^*(t)\}, & \theta(t) = 1 \end{cases} \tag{2.9}$$

$$\theta(t) = \varphi(t, y^*(t) - g(t, x(t)))$$

Equation (2.9) may be derived through schemes of dynamic programming (see [25]). Its solution should be taken in the generalized sense as mentioned above.

Theorem 2.2. If for $u = u^*(t), y = y^*(t)$ $(t \in [t_0, \tau])$ Eq. (2.9) has a generalized (viscosity) solution $V^{(1)}(t,x)$, then the following equality holds

$$\mathcal{X}(\tau,\cdot) = \{x : V^{(1)}(\tau,x) \le 1\} \tag{2.10}$$

In [26, 27] an answer to the question of "which evolution equation would the multivalued mapping $\mathcal{H}[t] = \mathcal{H}(t,\cdot) = \mathcal{H}(t,\cdot\,|\,\mathcal{H}(t_0,\cdot))$ satisfy?" was given; it was suggested that this evolution equation should be taken as the integral funnel equation for the differential inclusion

$$\frac{dx}{dt} \in f_1(t,x,u^*(t)) + f_2(t,x,\mathcal{Q}(t)) \tag{2.11}$$

with phase constraint

$$x(t) \in \mathcal{X}_Y(t) = \{x : g(t,x) \in Y^*(t) = y^*(t) - \mathcal{R}(t)\} \tag{2.12}$$

One such equation is [17] (see also [27])

$$\lim_{\sigma \to +0} \frac{1}{\sigma} h_+(\mathcal{Z}[t+\sigma], \cup\{x + \sigma(f_1(t,x,u^*(t)) + f_2(t,x,\mathcal{Q}(t)))\,|\,x \in \mathcal{Z}(t) \cap \mathcal{X}_Y(t)\}) = 0, \quad \mathcal{Z}(t_0) = \mathcal{X}^0 \tag{2.13}$$

where h_+ is the Hausdorff semidistance

$$h_+(\mathcal{X}', \mathcal{X}'') = \min\{\varepsilon : \mathcal{X}' \subseteq \mathcal{X}'' + \varepsilon\mathcal{B}(0)\}$$

and $\mathcal{B}(0)$ is the unit ball in $\mathbf{R}^n$. The solution $\mathcal{Z}[t]$ of Eq. (2.12) is a set-valued function, with $\mathcal{Z}[t_0] = \mathcal{H}^0$, and, as a rule, the solution is non-unique. The required solution $\mathcal{H}[t]$, which is identical with the realization of the function $\mathcal{H}(t,\cdot)$, is the inclusion-maximal solution of Eq. (2.12), namely, $\mathcal{H}[t] \supset \mathcal{Z}[t]$, where $\mathcal{Z}[t]$ is any solution which starts from $\mathcal{Z}(t_0) = \mathcal{H}^0$ [17]. Note that Eq. (2.12) makes sense for any piece wise-continuous functions $y^*(t)$. As mentioned above, we further presume that these functions are right-continuous.

Henceforth, it is always assumed that the functions $V(\tau,x)$ are continuous in all the variables, with non-empty compact zero level-sets

$$\mathcal{X}[\tau] = \{x : V(\tau,x) \le 0\} \ne \varnothing$$

obtained by virtue of equality (2.6). These are necessarily those that solve our problem for linear systems (1.1)–(1.3) with continuous coefficients and convex constraints, continuous in time. This class of functions $V(\tau,\cdot)$ will be denoted by $\mathcal{K}_V$.

Remark 2.3. The set $\mathcal{K}_V$ may be considered as a metric space with Hausdorff metric

$$d(V'(\tau,\cdot), V''(\tau,\cdot)) = h(\mathcal{X}', \mathcal{X}'')$$

where $\mathcal{H}'$ and $\mathcal{H}''$ are zero level-sets of the functions V' and V'', and the Hausdorff distance

$$h(\mathcal{X}, \mathcal{Z}) = \max\{h_+(\mathcal{X}, \mathcal{Z}), h_+(\mathcal{Z}, \mathcal{X})\}$$

where $h_+(\mathcal{H}, \mathcal{Z})$ is the Hausdorff semidistance.

The set $\mathcal{K}_V$ may be considered in another metric, introducing $d(V'(\tau, \cdot), V''(\tau, \cdot))$ as the distance in the space $C_r[t_0, \tau]$ (of r-dimensional continuous functions) between respective functions $y'(t)$ and $y''(t)$ ($t \in [t_0, \tau]$), which generated $V'(\tau, \cdot)$ and $V''(\tau, \cdot)$ due to Eqs (2.4) and (2.5). Such a definition may be extended only to functions $y(\cdot) \in Y(\cdot)$. (When applying this metric to any piecewise-continuous functions in the case when not every pair y', y'' has a corresponding element $V \in \mathcal{K}_V$, the distance $d(y', y'')$ may be taken as $+\infty$.)

Similar metrics may also be considered on the set of functions $V^{(1)}(\tau, x)$. A detailed discussion of possible metrics for the spaces of functions similar to those generated by the value functions of this paper, is given in monograph [12] (Chapter 4, Appendix C.5).

We will now consider the control synthesis problem for systems described by Eq. (2.7) or (2.9).

3. CONTROL SYNTHESIS FOR SYSTEMS WITH SET-VALUED SOLUTIONS

Thus, consider Eqs (2.4) and (2.5), which describe the dynamics of value function $V(t, x)$, whose level-sets are the estimates $\mathcal{H}[t]$ for the actual state of the overall system (1.1), (1.2).

Problem C. Find the value functional

$$\mathcal{V}(\tau, V(\tau, \cdot)) = \min_u \max_y \max_x \left\{ -\int_\tau^{t_1} d^2(y(t) - g(t, x(t)), \mathcal{R}(t))dt + \right.$$

$$\left. + d^2(x[t_1], \mathcal{M}) | u \in \mathcal{U}, y(\cdot) \in \mathcal{Y}(\cdot), V(\tau, x) \le 0 \right\} \tag{3.1}$$

with given element $V(\tau, \cdot) \in \mathcal{K}_V$.

Here the minimum is taken in the class $\mathcal{U}$ of positional strategies, $u = u(t, V)$, specified below. The maximum is taken over all $x \in \mathcal{X}[\tau] = \{x: V(\tau, x) \le 0\}$ and all $y \in \mathcal{Y}(t)$, where $\mathcal{Y}(t)$ is the set of all possible future realizations of the measurement $y(t)$ over the interval $(\tau, t_1]$. The set-valued function

$$\mathcal{Y}(t) = gt, X_-(t, \tau, \mathcal{X}[\tau] | u[\cdot])) + \mathcal{R}(t)$$

where $X_-(t, \tau, \mathcal{H}[\tau] \mid u[\cdot])$ is the reachability set of system (1.1) over the variable υ for a fixed realization $u[\cdot]$; $x[t] = x(t, \tau, x)$ ($\tau \le t \le t_1$), is the trajectory of system (1.1), driven *forward* from the point $\{\tau, x\}$.

Remark 3.1. Functional (3.1) relates to the second part of functional (1.5), which corresponds to the interval $(\tau, t_1]$ (see Remark 2.1).

The function $V(\tau, \cdot)$ (the information state of the system) evolves due to Eq. (2.4) with condition (2.5), for realization $u^*(t)$ and $y^*(t)$ given over the interval $t \in [t_0, \tau)$.

The meaning of functional $\mathcal{V}$ consists of the fact that when $\tau = t_1$ it ensures the condition (see relation (3.1))

$$\mathcal{V}(t_1, V(t_1, \cdot)) = \max\{d^2(x, \mathcal{M}) \mid V(t_1, x) \le 0\}$$

(The number $\gamma = \mathcal{V}(t_1, V(t_1, \cdot))$ is the size of the neighbourhood $\mathcal{M}_\gamma = \mathcal{M} + \gamma\mathcal{B}(0)$, which at time t_1 entirely includes $\mathcal{H}[t_1]$, so that $\mathcal{H}[t_1] \subseteq \mathcal{M}_\gamma$).

Hence it can be seen that the boundary condition for Problem C is taken at the right end of the interval $[\tau, t_1]$. It has the form

$$\mathcal{V}(t_1, V(t_1, \cdot)) = \max_x\{d^2(x, \mathcal{M}) \mid V(t_1, x) \le 0\} \tag{3.2}$$

where $V(t_1, \cdot) \in \mathcal{K}_V$.

Remark 3.2. For the compactum $\mathcal{X}$ the following equality holds

$$\max\{d^2(x, \mathcal{M}) \mid x \in \mathcal{X}\} = h_+^2(\mathcal{X}, \mathcal{M})$$

The functional $\mathcal{V}(\tau, V(\tau, \cdot))$ is defined on the product of the spaces $\mathbf{R}_+ \times \mathcal{K}_V$, where $\mathbf{R}_+ = [t_0, \infty)$. We introduce the following additional notation for this functional

$$\mathcal{V}(\tau, V(\tau, \cdot)) = \mathcal{V}(\tau, V(\tau, \cdot) \mid t_1, \mathcal{V}(t_1, \cdot)) \tag{3.3}$$

emphasizing its dependence on the boundary value $\mathcal{V}(t_1, \cdot)$.

We recall that the state of the system is taken here precisely as the pair $\{t, V(t, \cdot)\}$. The equation of further "motion" for $t \in [\tau, t_1]$ will now be the evolution equation

$$\frac{\partial V(t, \cdot)}{\partial t} = \Phi(t, V_x(t, \cdot), \cdot \mid u, y) \tag{3.4}$$

where, starting from the instant $t = \tau$ and later, the synthesized control strategy $u = u(t, V(t, \cdot))$ will be selected during the course of the process, depending on the known realization of the state of the system $\{t, V(t, \cdot)\}$, while the new measurement $y(t)$ will arrive throughout the course of the process. (Due to the last explanation, unlike Eq. (2.7), here u and y are not marked with asterisks.) Note that the nonlinear control strategy $u = u(t, V(t, \cdot))$ should be selected within the class of functionals which ensure the existence, in some reasonable sense, of the solution of Eq. (3.4) taken with this control strategy.

Thus the equation of dynamics for the overall system under incomplete measurements has the form (3.4), where the control u is the same as before, while the set of uncertain items, determined by the triple $\zeta_\tau = \{x \in \mathcal{H}[\tau], \upsilon(t) \in \mathcal{Q}(t), \xi(t) \in \mathcal{R}(t);\ t \in [\tau, t_1]\}$, is represented by the pair $\{\mathcal{H}[\tau], y(t);\ t \in [\tau, t_1]\}$, which had absorbed all the available current information on these parameters.

Lemma 3.1. The following relation, which is the principle of optimality in the class of system states $\{t, V(t, \cdot)\}$, $\tau \le t \le t_1$, holds

$$\mathcal{V}(\tau, V(\tau, \cdot) | t_1, \mathcal{V}(t_1, \cdot)) = \mathcal{V}(\tau, V(\tau, \cdot) | t, \mathcal{V}(t, \cdot \mid t_1, \mathcal{V}(t_1, \cdot)))$$

The value $\mathcal{V}(t_1, \cdot)$ is determined by the boundary condition, for example, by (3.2).

Following the schemes of dynamic programming, the last relation yields a generalized equation of the Hamilton–Jacobi–Bellman–Isaacs (HJBI) type, which is well known for game-theoretic systems with complete information [5, 10, 28, 29]. This equation may be formally written as

$$\min_u \max_y \left\{ \frac{d\mathcal{V}(\tau, V(\tau, \cdot))}{d\tau} \,\middle|\, u \in \mathcal{P}(\tau), y \in \mathcal{Y}(\tau) \right\} = 0 \tag{3.5}$$

Here $d\mathcal{V}(\tau, V(\tau, \cdot))/d\tau$ is the generalized total derivative of the functional $\mathcal{V}(t, V(t, \cdot))$ due to evolution equation (3.4).

Equation (3.5) may have a smooth solution, when the functional partial derivatives involved are understood in the strong or weak sense and the equation holds everywhere. (Such situations arise in linear systems, which are treated in the next section.) However, when there is no smoothness, it makes sense, for example, to consider generalized directional derivatives in the sense of Clarke of the functional $\mathcal{V}(\tau, V)$ along directions $\eta\{u(\cdot), y(\cdot)\}$, $t \in [\tau, \tau + \sigma]$. These derivatives determine the increments

$$\eta(\cdot) = V(t + \sigma, \cdot \mid u(\cdot), y(\cdot)) - V(t, \cdot), \quad \sigma > 0$$

(see [24]) in the limit relation

$$\partial_+\mathcal{V}[\tau, \cdot \mid \eta] = \lim_{\delta \to +0} \frac{1}{\delta} \sup_{V' \to V} (\mathcal{V}(\tau + \delta, V'(\tau + \delta, \cdot) + (\tau + \delta)\eta(\cdot)) - \mathcal{V}(\tau, V'(\tau, \cdot)))$$

where the convergence of V' and V is considered in the sense of the distances $d(V, V')$ introduced earlier (see Remark 2.3).

Remark 3.3. Note that it is also possible to interpret the generalized solutions of the present situation in the earlier "viscosity" sense [20, 21] (see also [23]). However, due to the specific structure of the present infinite-dimensional

Problem C, the respective definitions require special modification which takes us beyond the scope of the present paper. A description of generalized solutions of the HJB equation in infinite-dimensional space is given in [12, 24].

Strategy $u^0(t, V(t, \cdot))$, obtained from minimization condition in relation (3.5), has the form

$$u^0 = u^0(t, V(t, \cdot)) \subseteq \mathcal{P}(t)$$

Substituting this into (2.7), we arrive at the formal equation

$$\frac{\partial V}{\partial t} + \max_{v}\left\{\left(\frac{\partial V}{\partial x}, f_1(t, x, u^0(t, V(t, \cdot))) + f_2(t, x, v)\right)\middle| v(t) \in \mathcal{Q}(t)\right\} - \\ - d^2(y(t) - g(t, x), \mathcal{R}(t)) = 0 \tag{3.6}$$

The last equation can be considered as an evolution equation in a metric space $\mathcal{H}_V$ of the functions $\{V(t, \cdot)\}$.

The solutions of this equation, the "trajectories" $V[t] = V(t, \cdot)$, issuing at a time t_0 from $V(t_0, \cdot) = d^2(x, \mathcal{X}^0)$, may then be interpreted as "constructive motions" $V(t, \cdot; u^0(t, \cdot))$, obtained as a result of limit transition from infinite-dimensional analogues of Euler's broken lines, constructed under all possible partitions of the interval $[t_0, t_1]$, by selecting piecewise-constant realizations $u[t] \in \mathcal{P}(t)$ [5, p. 11]. This is the class of strategies $\mathcal{U}$ suggested in general. Note that strategies $u(t, V(t, \cdot)) \in \mathcal{U}$ do not generally ensure the uniqueness of the solution of system (3.6).

Theorem 3.1. The control strategy $u = u^0(t, V(t, \cdot)) \in \mathcal{U}$ guarantees the condition

$$\max_y \max_V\left\{\max_x\{d^2(x, \mathcal{M}) \mid V(t_1, x | V(\tau, \cdot)) \le 0\} \middle| u = u^0(t, V(t, \cdot)), y(\cdot) \in \mathcal{Y}(\cdot)\right\} \le \\ \le \max_y \max_V\left\{\max_x\{d^2(x, \mathcal{M}) \mid V(t_1, x | V(\tau, \cdot)) \le 0\} \middle| u = \mathcal{U};\ y(\cdot) \in \mathcal{Y}(\cdot)\right\} \tag{3.7}$$

compared with any other strategy $u = u(t, V(t, \cdot)) \in \mathcal{U}$.

Here, due to the non-uniqueness of the solution of Eq. (3.6), the maximum in V is taken over all solutions $V(t, \cdot)$ of system (3.6), for any given strategy $u^0(t, V(t, \cdot)) \in \mathcal{U}$.

Remark 3.4. The above approach to Problems E and C, based on a transition to value functions for dynamic programming problems, essentially indicates that the overall Problem $E - C$ of measurement feedback control synthesis allows of a certain "separation principle", namely, independent solution of Problems E and C. Here Problem E (of estimation) turns out to be finite-dimensional, while Problem C (of control) is infinite-dimensional. The last comment is of interest in specifying special cases when Problem C may also be reduced to one of finite dimension.

The detailed procedure for calculating controls using the scheme suggested requires, in general, a more detailed elaboration. The scheme becomes simpler when the system is linear.

In the linear case ($f_1 = Ax + Bu, f_2 = Cv, g(t, x) = Hx$), treated below, the class $\mathcal{U}$ of accepted synthesized strategies will consist of set-valued functionals $u^0(t, V)$ and $u^0(t, \mathcal{X})$, upper semicontinuous with respect to inclusion in the respective metric. The synthesized system (3.6) will then turn into a functional-differential inclusion of the type

$$\frac{\partial V}{\partial t} \in -\max_{v}\left\{\left(\frac{\partial V}{\partial x}, f_1(t, x, u^0(t, V(t, \cdot)) + f_2(t, x, v)\right)\middle| v(t) \in \mathcal{Q}(t)\right\} - \\ - d^2(y^*(t) - g(t, x), \mathcal{R}(t)) \tag{3.8}$$

We will now present a more detailed discussion of Problems E and C for a linear system.

4. LINEAR SYSTEMS

Suppose the original system is of the form

$$\dot{x} = A(t)x + B(t)u + C(t)v \tag{4.1}$$

and the equation of observations is also linear

$$y(t) = H(t)x + \xi(t) \tag{4.2}$$

Here the constraints on u, υ and ξ are defined using continuous set-valued functions $\mathcal{P}(t)$, $\mathcal{Q}(t)$ and $\mathcal{R}(t)$, whose values are subsets of non-degenerate ellipsoids $\mathcal{E}(p(t), P(t))$, $\mathcal{E}(q(t), Q(t))$ and $\mathcal{E}(r(t), R(t))$ of dimensions p, q and r respectively. The $r \times n$ matrix H, the non-singular "configuration matrices" $P(t)$, $Q(t)$ and $R(t)$ and the centres $p(t)$, $q(t)$ and $r(t)$ of given ellipsoids are assumed continuous.

Here we use the rotation

$$\mathcal{E}(p, P) = \{u : (u - p, P^{-1}(u - p)) \le 1\}$$

The initial and objective sets, $\mathcal{X}^0$ and $\mathcal{M}$, are also taken as non-degenerate ellipsoids: $\mathcal{E}(x_0, X_0)$ and $\mathcal{E}(m, M)$.

In this case the corresponding value functions can be computed using convex analysis techniques [14, 30, 31]. We will discuss the corresponding solutions.

Consider Problem E_0. We start by solving an equation of the form (3.6) – the "forward" HJB equation for the linear case. We have

$$\frac{\partial V}{\partial t} + \max_{\upsilon}\left\{\left(\frac{\partial V}{\partial x}, A(t)x + B(t)u^* + C(t)\upsilon\right) + d^2(y^*(t) - H(t)x, \mathcal{R}(t)) \,|\, \upsilon(t) \in \mathcal{Q}(t)\right\} = 0 \tag{4.3}$$

under condition (2.5). To do this we fix the known realizations of the observation $y^*(t)$ and the control $u = u^*(t)$, $t \in [t_0, \tau]$, and consider the relation

$$V(\tau, x) = \min_{\upsilon}\left\{d^2(x(t_0), X^0) + \int_{t_0}^{\tau} d^2(y^*(t) - H(t)x(t), \mathcal{R}(t))dt \,|\, \upsilon(t) \in \mathcal{Q}(t), x(\tau) = x\right\} \tag{4.4}$$

Then the information set $\mathcal{X}[\tau]$ can be expressed as a level set of (2.2). It can be obtained by solving the optimization problem of finding the function $V(\tau, x)$, which may be calculated using convex analysis.

Let $\rho(l \,|\, \mathcal{X}) = \max\{(l, x) \,|\, x \in \mathcal{X}\}$ denote the support function of the compactum $\mathcal{X}$ along the direction l. We use the relation

$$d^2(x(t_0), X^0) = \max\left\{(l, x(t_0)) - \rho(l|X^0) - \frac{1}{4}(l, l) \,\middle|\, l \in \mathbf{R}^n\right\} \tag{4.5}$$

as well as

$$\int_{t_0}^{\tau} d^2(y^*(t) - H(t)x(t), \mathcal{R}(t))d\alpha(t) =$$

$$= \max\left\{\int_{t_0}^{\tau}\left((\lambda(t), y^*(t) - H(t)x(t)) - \rho(\lambda(t)|\mathcal{R}(t)) - \frac{1}{4}(\lambda(t), \lambda(t))\right)d\alpha(t) \,\middle|\, \lambda(\cdot) \in C_r[t_0, \tau]\right\}$$

for any $\alpha(\cdot) \in \mathrm{Var}_+[t_0, \tau]$, where, following [31], $C_r[t_0, \tau]$ is the space of r-dimensional continuous functions and $\mathrm{Var}_+[t_0, \tau]$ is the class of non-decreasing functions of unit variation. We obtain

$$\begin{aligned} V(\tau, x) = \max\Big\{&(s(\tau), x) - \int_{t_0}^{\tau}\rho(s(t)|C(t)\mathcal{Q}(t))dt + \int_{t_0}^{\tau}((\lambda(t), y^*(t)) - (s(t), B(t)u^*(t)) - \\ &- \rho(\lambda(t)|\mathcal{R}(t)))d\alpha(t) - \rho(l|\mathcal{X}^0) - \frac{1}{4}(l, l) - \frac{1}{4}\int_{t_0}^{\tau}(\lambda(t), \lambda(t))d\alpha(t)\Big\} \end{aligned} \tag{4.6}$$

where $s(t)$ is the solution to the adjoint system of Problem E_0, i.e. of the equation

$$ds = -sA(t)dt + \lambda'(t)H(t)d\alpha(t), \quad s(t_0) = l$$

Note that henceforth the functions $y^*(t)$ are assumed to be left-continuous, while $\alpha(t)$ is assumed to be right-continuous.

In problem (4.6) maximum are reached and are unique. We will denote the corresponding maximizers by $\{l^0, \lambda^0(\cdot), \alpha^0(t)\}$, and the solution $s(t)$ of the adjoint equation, obtained by substituting these maximizers, by $s^0(t)$. After substituting $\{l^0, \lambda^0(\cdot), \alpha^0(t), s^0(t)\}$ into (4.6) and using the rules for differentiating maximum functions (see [32]), we obtain

$$V_\tau(\tau, x) = (\dot{s}^0(\tau), x) + (\lambda^0(\tau), y^*(\tau)) -$$
$$-\rho(s^0(\tau)C(\tau)|\mathcal{Q}(\tau)) - (s^0(\tau), B(\tau)u^*(\tau)) - \rho(\lambda^0(\tau)|\mathcal{R}(\tau)) - \frac{1}{4}(\lambda^0(\tau), \lambda^0(\tau))$$
$$V_x(\tau, x) = s^0(\tau)$$

Here we have assumed that at a time τ the function $\alpha^0(t)$ has no jump. After substituting the resulting expressions into Eq. (4.3) we see that $V(t, x)$ satisfies this equation everywhere.

When $t = t_0$ the function $V(t, x)$ (4.6) leads to an expression identical with boundary condition (2.5).

It is possible to check by direct substitution that this result holds even if the function $\alpha^0(t)$ has a jump at time τ.

Therefore, we have the following lemma.

Lemma 4.1. The function $V(t, x)$ (4.6) satisfies Eq. (4.3) everywhere, and also boundary condition (2.5).

Knowing $V(\tau, x)$ it is possible to compute

$$\rho(l|\mathcal{X}[\tau]) = \max\{(l, x)|V(\tau, x) \le 0\}$$

(This function can also be computed directly as indicated above, see [14, p. 94].)

Namely

$$\rho(l|\mathcal{X}[\tau]) =$$
$$= \min_{\Lambda}\left\{\rho(\psi(t_0)|\mathcal{X}^0) + \int_{t_0}^{\tau}(\rho(\psi(t)|C(t)\mathcal{Q}(t)) + B(t)u(t))d\Lambda(s) + \int_{t_0}^{\tau}\rho(d\Lambda(s)| - Y(s))\right\} \tag{4.7}$$

where $Y(t) = y(t) - \mathcal{R}(t)$, $\Lambda(t) \in \mathrm{Var}_r[t_0, \tau]$ is an r-dimensional function of bounded variation over $[t_0, \tau]$. Here $\psi(t)$ is the solution of the adjoint equation in the second form

$$d\psi = -\psi A(t)dt - d\Lambda(t)H(t), \quad \psi(\tau) = l'$$

We will now consider Problem C of control synthesis, namely, we shall seek a solution of the following problem: it is required to find

$$\mathcal{V}(\tau, \mathcal{X}[\tau]) = \min_u \max_y \left\{\max_x\{d^2(x, \mathcal{M})|x \in \mathcal{X}[t_1]\}\middle|u \in \mathcal{U}_V, y(\cdot) \in \mathcal{Y}(\cdot)\right\} \tag{4.8}$$

or, which is the same

$$\mathcal{V}(t, V) = \min_u \max_y \left\{\max_x\{d^2(x, \mathcal{M})|\mathcal{V}(t_1, x) \in 0\}\middle|u \in \mathcal{U}_V, y(\cdot) \in \mathcal{Y}(\cdot)\right\}$$

since $\mathcal{V}(t, \mathcal{H}[\tau]) = \mathcal{V}(t, V(t, \cdot))$.

Here

$$\mathcal{X}[t_1] = \mathcal{X}(t_1, \tau, \mathcal{X}[\tau]), \quad \mathcal{X}[t_1] = \{x : V(t_1, x) \le 0\}$$

and $\mathcal{U}_V$ is the class of feedback control strategies of the form $\mathcal{U}(t, V) \in \mathcal{P}(t)$ with values in the class of convex compacta of the space $\mathbf{R}^p$. The functionals $\mathcal{U}(t, V)$ are defined for every t on the set $\mathcal{K}_V$ of convex functions $V(x)$ with non-empty compact level sets

$$\mathcal{X}_V = \{x : V(x) \le 0\} \neq \varnothing$$

The sets $\mathcal{H}_V$ are compact and convex. We will consider these sets as elements of the corresponding metric space $\mathcal{H}_X$ with Hausdorff metric. The class $\mathcal{U}_V$ consists of strategies of the form $\mathcal{U}(t, V)$, upper semicontinuous in $V \in \mathcal{H}_V$ by virtue of the metric introduced, and continuous in t. The functions $V(t,x)$ and hence the sets $\mathcal{X}[t]$ evolve according to Eq. (4.3), where the right derivatives of the functional $\mathcal{V}(\tau, \mathcal{H}[\tau])$ are computed through values $u(\tau) = u(\tau + 0)$ and $y(\tau) = y(\tau + 0)$, to be realized.

It view of the directional differentiability of the functional $\mathcal{V}(t, V)$ Eq. (3.5) takes the form

$$\min_u \max_y \max_V \left\{ \frac{d\mathcal{V}(\tau, V(\tau, \cdot))}{d\tau} \middle| u \in \mathcal{P}(\tau), y \in \mathcal{Y}(\tau) = H(\tau)\mathcal{X}[\tau] + \mathcal{R}(\tau) \right\} = 0 \tag{4.9}$$

The maximum over V is taken due to the non-uniqueness of the solution of the linear version of differential inclusion (3.8). The boundary condition is given by equality (3.3).

Here $d\mathcal{V}(\tau, V)/d\tau$ is the total derivative of the functional $\mathcal{V}(\tau, V)$ with respect to system (4.3).

The above mentioned "backward" HJB equation (4.9) in the state space of $\{\tau, \mathcal{X}[\tau]\}$ is similar. Note straightaway that due to the linearity of the original system the min and max operations in (4.9) are interchangeable.

The solution of Problem C can thus be obtained by solving Eq. (4.6). However, in the case considered one can avoid finding such a solution by using "aiming" techniques similar to those introduced by N. N. Krasovskii [4, 5, 33]. With this in view it is necessary to construct an analogue of the stable "Krasovskii bridge" in the corresponding infinite-dimensional space $\mathcal{H}_V$ or $\mathcal{H}_X$. We will do this.

Consider a uniform partition $\Sigma(N)$ of the interval $[\tau, t_1]$ into N intervals by points $\tau + \sigma_N, \ldots, t_1 - \sigma_N, t_1$; $\sigma_N = (t_1 - \tau)/N$. Put

$$\tau_N^{(k)} = \tau + k\sigma_N, \quad k = 1, \ldots, N-1$$

For this partition we solve a sequence of open-loop control problems. For $k = 1$ we have

$$\max_y \min_u \{(\max\{d^2(x, \mathcal{M}) | V(t_1, x | t_1 - \sigma_N, V(\cdot)) \le 0\} | u(\cdot) \in \mathcal{P}(\cdot), y(\cdot) \in \mathcal{Y}(\cdot),$$
$$V(\cdot) = V(t_1 - \sigma_N, \cdot)\} = \mathcal{V}_N^{(1)}(t_1 - \sigma_N, V(t_1 - \sigma_N, \cdot))$$

or, by introducing the notation

$$\mathcal{V}(t_1, V(t_1, \cdot)) = \max\{d^2(x, \mathcal{M}) | V(t_1, x) \le 0\}$$

we obtain in operator form

$$T_N^{(1)}\mathcal{V}(t_1, V(t_1, \cdot)) = \mathcal{V}_N^{(1)}(t_1 - \sigma_N, V(t_1 - \sigma_N, \cdot))$$

Further, for $k = 2$ we get

$$\max_y \min_u \{\mathcal{V}(t_1 - \sigma_N, V(\cdot) | u(\cdot), y(\cdot), V(\cdot) = V(t_1 - 2\sigma_N, \cdot))\} =$$
$$= \mathcal{V}_N^{(2)}(t_1 - 2\sigma_N, V(t_1 - 2\sigma_N, \cdot)), \ldots$$

or in operator form

$$T_N^{(2)}\mathcal{V}_N^{(1)}(t_1 - \sigma_N, V(t_1 - \sigma_N, \cdot)) = \mathcal{V}_N^{(2)}(t_1 - 2\sigma_N, V(t_1 - 2\sigma_N, \cdot))$$

Continuing this process from $k = 3$ to $k = N$, we obtain the equality

$$\max_y \min_u \{\mathcal{V}(\tau + \sigma_N, V(\cdot) | u(\cdot), y(\cdot), V(\cdot) = V(\tau, \cdot)\} =$$
$$= T_N^{(N)}\mathcal{V}_N^{(N-1)}(\tau + \sigma_N, V(\tau + \sigma_N, \cdot)) = \mathcal{V}_N^{(N)}(\tau, V(\tau, \cdot))$$

or

$$\mathcal{V}_N^{(N)}(\tau, V(\tau, \cdot)) = T_N^{(N)} \cdot T_N^{(N-1)}, \ldots, T_N^{(2)} \cdot T_N^{(1)}\mathcal{V}(t_1, V(t_1, \cdot)) = \mathcal{T}_N\mathcal{V}(t_1, V(t_1, \cdot)) \tag{4.10}$$

Similar constructions are also possible with non-uniform partitions of the interval $[\tau, t_1]$.

The sequence of partitions $\Sigma(N)$, which arise as N increases, will be called *monotone*, if new partitions $\Sigma(N_1)$, with $N_1 > N$, are obtained by adding new points to the previous one.

Lemma 4.2. If the sequence $\Sigma(N)$ is monotone as N increases, and the integer $N \leq M$, then the following inequalities hold

$$\mathcal{V}_N(\tau, V(\tau, \cdot)) \leq \mathcal{V}_M(\tau, V(\tau, \cdot))$$

Considering the classes $\mathcal{W}_N^{(k)}$ of convex compact subsets $W \in \mathcal{K}_X$, which satisfy the condition

$$\{W : \mathcal{V}_N^{(k)}(\tau - k\sigma_N, W) \leq 0\} = \mathcal{W}_N^{(k)}[\tau - k\sigma_N]$$

we further put

$$\mathcal{V}_N^{(N)}(\tau, \cdot) = \mathcal{V}_N, \quad \mathcal{W}_N^{(N)}(\tau) = \mathcal{W}_N$$

The following condition follows from Lemma 4.2.

Lemma 4.3. If the sequence $\Sigma(N)$ is monotone as N increases, and the integer $N \leq M$, then the inclusions $\mathcal{W}_M[\tau] \subseteq \mathcal{W}_N[\tau]$ hold.

Recall that the diameter of the (non-empty) compact set $\mathcal{Z}$ is taken as the number

$$D(\mathcal{Z}) = \max_l \{\rho(l|\mathcal{Z}) + \rho(-l|\mathcal{Z}) | (l, l) = 1\}$$

Assumption 4.1. For any function $y(t) \in \mathcal{Y}(t)$ ($t \in [\tau, t_1]$), there exist numbers N_0, $\varepsilon > 0$, and a continuous multivalued function $\mathcal{Z}(t)$, with values in $\mathcal{K}_X$, of diameter $D(\mathcal{Z}) \leq \varepsilon$, such that the inclusions $\mathcal{Z}(\tau_N^k) \subseteq \mathcal{W}_N^{(k)}$ are true, whatever the numbers $N \geq N_0$, $k \in [1, \dots, N-1]$ and corresponding sets $\mathcal{W}_N^{(k)}$ constructed with the function $y(t)$.

Assumption 4.1 will further be considered to be true. Then, increasing the partition $\Sigma(N)$ and taking the limit as $N \to \infty$ ($\sigma_N \to +0$), we obtain from the previous lemma a non-decreasing sequence $\mathcal{V}_N(\tau, V(\tau, \cdot)))$, bounded from above for every $V(\tau, \cdot)$ (a similar case is presented in [34]), which hence converges "pointwise" to some functional $\mathcal{V}(\tau, V(\tau, \cdot))$. This functional is continuous in t and V (in the metric considered).

In other words, as $N \to \infty$ ($\sigma_N \to +0$), there is a "pointwise" convergence

$$\lim \mathcal{T}_N \mathcal{V}(t_1, V(t_1, \cdot)) = \mathcal{T}\mathcal{V}(t_1, V(t_1, \cdot)) = \mathcal{V}(\tau, V(\tau, \cdot))$$

Here the operators $\mathcal{T}_N$ are the analogues of alternated integral sums, and the operator $\mathcal{T}$ is a certain multivalued integral, which is an analogue of *Pontryagin's Alternated Integral* [35], but constructed in an infinite-dimensional space. It does not depend on the method of partitioning the interval $[\tau, t_1]$ when constructing the integral sums.

We introduce a class of subsets W of space $\mathcal{K}_X$

$$\mathcal{W}[\tau] = \{W : \mathcal{V}(\tau, W) \leq 0 | W \in \mathcal{K}_X\} \tag{4.11}$$

Note that the union

$$\mathbf{W}[\tau] = \cup\{x : x \in W | W \in \mathcal{W}[\tau]\}$$

is convex and compact in $\mathbf{R}^n$. This can be verified by direct reasoning.

The set-valued function $\mathcal{W}[t]$ (with values in the set of subsets of $\mathcal{K}_X$) will be used further to construct the desired synthesizing control strategy $\mathcal{U}(t, V)$. This function is an analogue of the "Krasovskii bridge" in the space $\mathcal{K}|\mathcal{W}|$ of set $\mathcal{W}$ of convex compact subsets W of the space $\mathcal{K}_X$.

Let us find the control strategy $\mathcal{U}(t, V(t, \cdot))$ which solves Problem C. Consider the following subproblem: it is required to find the semidistance

$$h_+(\mathcal{X}[\tau], \mathcal{W}[\tau]) = \min\{h_+(\mathcal{X}[\tau], W) | W \in \mathcal{W}[\tau]\} = \varepsilon(\tau, \mathcal{X}[\tau])$$

Calculating it we have

$$\varepsilon(\tau, \mathscr{X}[\tau]) = \max\{\rho(l|\mathscr{X}[\tau]) - \rho_W(l, \tau) | (l, l) \le 1\} \tag{4.12}$$
$$\rho_W(l, \tau) = \max\{\rho(l|W) | W \in \mathscr{W}[\tau]\}$$

Having checked that the equality $\rho_W(l, \tau) = \rho(l|\mathbf{W}[\tau])$ is true, we note that the maximizer $l^0 = l^0(\tau, \mathscr{H}[\tau])$ in problem (4.12) is unique. Let $W^0[\tau] \in \mathscr{W}[\tau]$ be the maximum with respect to inclusion among the subsets $W \in \mathscr{W}[\tau]$ on which the following equality holds

$$\rho(l^0|W^0[\tau]) = \rho_W(l^0, \tau) = \rho(l^0|\mathbf{W}[\tau])$$

We compute the total derivative of the functional $\varepsilon(\tau, \mathscr{H}[\tau])$ along the "trajectories" of system (4.3), (2.5) by applying the rules of differentiating a maximum function, when the vector l^0 is unique, along the direction

$$u = u', \quad y = y'(\tau + 0) = H(\tau)x'[\tau] + \xi'[\tau]$$

Assuming further that $\varepsilon(\tau, \mathscr{H}[\tau]) > 0$, we obtain

$$\left.\frac{d\varepsilon(\tau, \mathscr{X}[\tau])}{d\tau}\right|_{u, y} = \left.\frac{\partial\rho(l^0|\mathscr{X}[\tau])}{\partial\tau}\right|_{u, y} - \frac{\partial\rho(l^0|W^0[\tau])}{\partial\tau} \tag{4.13}$$

where

$$\left.\frac{\partial\rho(l^0|\mathscr{X}[\tau])}{\partial\tau}\right|_{u, y} =$$
$$= \rho(l^0(\tau)|C(\tau)(\mathscr{Q}(\tau) - v'(\tau)) + \rho(\lambda^0[\tau]|\mathscr{R}(\tau + 0) - \xi'(\tau + 0)) + (l^0, B(\tau)u)$$

and $\lambda^0[\tau]$ is a jump at the instant $\tau + 0$ of the unique extremal element of problem (4.7), namely, of measure Λ^0 (here we employ the ellipsoidal nature of the constraints $\mathscr{H}^0$, $\mathscr{R}(t)$).

Relying on the representation of (4.10) and (4.11) in terms similar to (4.7) and (4.6), and omitting the detailed calculation, we have

$$\frac{\partial\rho(l^0|W^0[\tau + 0])}{\partial\tau} = -\rho(l^0|B(\tau)\mathscr{P}(\tau))$$

The derivatives obtained have to be further substituted into (4.13). We then maximize the resulting expression with respect to $\zeta[\tau]$ over all the "triples"

$$\zeta[\tau] = \{x' \in \mathscr{X}[\tau], v'(\tau + 0) \in \mathscr{Q}(\tau), \xi'(\tau + 0) \in \mathscr{R}(\tau)\}$$

(Since inspecting all $\zeta(t), t \in [\tau, \tau + \sigma]$ means inspecting all $y(t)$ in the same interval and all $x' \in \mathscr{H}[\tau]$, the operation of differentiation along the direction $y(t + 0)$ with subsequent maximization over $y(t + 0)$ is replaced with similar operations for ζ.)

As a result we have

$$\max_{\zeta}\left\{\left.\frac{\partial\rho(l^0, \mathscr{X}[\tau])}{\partial\tau}\right|_{u, \zeta} - \frac{\partial\rho(l^0|W^0[\tau + 0])}{\partial\tau}\right\} = -(-l^0, B(\tau)u) + \rho(l^0| - B(\tau)\mathscr{P}(\tau)) \tag{4.14}$$

whence the following statement follows.

Theorem 4.1. The inequality

$$\frac{d\varepsilon(\tau, \mathscr{X}[\tau])}{d\tau} \le 0$$

is true, whatever the realization $y(t)$ ($t \in [\tau, t_1]$), if and only if the strategy $u = \mathcal{U}^0(\tau, \mathcal{H}[\tau])$, or $u = \mathcal{U}^0(\tau, V(\tau, \cdot))$, which is the same, satisfies the maximum condition

$$\mathcal{U}^0(\tau, \mathcal{X}[\tau]) = \arg\max\{(-l^0, B(\tau)u) \mid u \in \mathcal{P}(\tau)\}$$

Integrating the derivative $d\varepsilon(t, \mathcal{H}[t])/dt$ from τ to t_1 with $u = \mathcal{U}^0(t, \mathcal{H}[t])$, we find

$$\varepsilon(t_1, V(t_1, \cdot)) = \varepsilon(t_1, \mathcal{X}[t_1]) \le \varepsilon(\tau, \mathcal{X}[t]) = \varepsilon(\tau, V(t, \cdot))$$

for any realization $y(t)$, $t \in [\tau, t_1]$.

Here we used the equalities

$$\varepsilon(t_1, \mathcal{X}[t_1]) = \max_x\{d(x, \mathcal{M}) \mid V(t_1, x) \le 0\}, \quad \mathcal{X}[\tau] = \{x : V(\tau, x) \le 0\}$$

As a consequence we arrive at the following statement.

Theorem 4.2. If the control process starts at time τ, then the strategy

$$u = \mathcal{U}^0(t, \mathcal{X}[t]) = \mathcal{U}^0(t, V(t, \cdot)), \quad t \ge \tau$$

guarantees the inequality

$$\varepsilon(t_1, V(t_1, \cdot)) = \max_x\{d(x, \mathcal{M}) \mid V(t_1, x) \le 0\} \le \varepsilon(\tau, V(\tau, \cdot))$$

despite the disturbances and the incompleteness of the measurements.

One may choose, of course, t_0 as the instant τ.

This completes the scheme of the simultaneous solution of Problems E and C. As indicated, these problems are divided into two parts: the problem of guaranteed positional estimation (observation) and the problem of guaranteed positional control (feedback synthesis). Namely, the following statement is true.

Theorem 4.3. The strategy $u = \mathcal{U}^0(t, \mathcal{H}[t])$ sought for Problem $E - C$ may be found by solving Problem C along the "trajectories" $V(t, \cdot)$ of Eq. (4.3), which solves Problem E. This strategy guarantees that the set-valued trajectory of system (1.1) is entirely included in an $\varepsilon(t_1, V(t_1, \cdot))$-neighbourhood of the objective set $\mathcal{M}$ despite the disturbances and the incompleteness of the measurements.

Concluding remark. Along with the development and justification of the theoretical details of the above approach, a pressing problem is computing the solutions, which in the non-linear case reduces to the development of methods for the rapid calculation of solutions of the HJB equations. Here the approaches of [36, 37] seem to be promising. In the linear case it is possible to find a solution based on relations of convex analysis. A numerical realization of such approaches may be carried out using the ellipsoidal calculus and related techniques [17, 38–41].

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